Pergamon

Int. J. Solids Structures Vol. 31, No. 7, pp. 953–970, 1994

0020–7683/94 $6.00 + .00

STEADY STATES AND SENSITIVITY ANALYSIS IN ELASTIC–PLASTIC STRUCTURES SUBJECTED TO CYCLIC LOADS

CASTRENZE POLIZZOTTO
Dipartimento di Ingegneria Strutturale & Geotecnica, DISEG, Università di Palermo,
90128 Palermo, Italy

(*Received* 27 *February* 1993; *in revised form* 6 *October* 1993)

Abstract—The steady-state response of an elastic–plastic structure subjected to quasi-static cyclic loads is investigated in the hypothesis of material models with dual internal variables and thermodynamic potential and for small displacements. The characters of the steady cycle are studied together with the related governing equations. With the aid of a sensitivity analysis with respect to the load parameters, a number of known results holding within perfect plasticity are extended to nonlinear hardening, and in particular certain properties of the structure in a condition of plastic shakedown are recognized to hold also in the presence of hardening materials. Perspectives for further developments conclude the paper.

1. INTRODUCTION

According to experimental evidence, a wide class of elastic–plastic and elastic–viscoplastic materials subjected to either stress-controlled or strain-controlled load cycles exhibit a steady cyclic behavior independent of the initial state (see e.g. Lemaitre and Chaboche, 1985). This material behavior induces one to conjecture that a structure made of such a material and subjected to a cyclic load manifests a similar behavior, at least as far as other nonlinearity sources (e.g. geometrical nonlinearities) remain negligible. Such material-to-structure extrapolation deserves, however, a direct proof for each material type not only for justifying it, but also for establishing the precise characters of the structure's stabilized response, as well as their dependence on the relevant (material, structural and loading) parameters. For a structure showing such a stabilized behavior under cyclic loading, two phases can be distinguished: (i) a short term transient response lasting in general a few cycles, which depends on the initial conditions and exhibits no periodicity features; and (ii) a long term stabilized (or steady-state) response exhibiting periodicity features independent of the initial conditions (*steady cycle*). Since in general the steady-state response phase covers almost the entire working life of the structure, knowing the steady cycle, possibly without the necessity of a full step-by-step analysis, represents a paramount research issue, e.g. for low-cycle fatigue failure criteria applications.

The existence of a steady cycle was proven by Frederick and Armstrong (1966) for elastic–perfectly plastic and elastic–perfectly viscoplastic materials, showing that the stresses and the plastic strain rates eventually become periodic with the same period of the loads (see also Ainsworth, 1977; Gokhfeld and Cherniavsky, 1980; Martin, 1975; Ponter, 1972). The above results were extended by Mróz (1972) and Ainsworth (1977) to a class of elastic–kinematically hardening viscoplastic materials, and by Halphen (1978) to elastic–plastic and elastic–viscoplastic standard materials with linear hardening.

Generalized standard materials, namely materials with dual internal variables and a convex thermodynamic potential (Halphen and Nguyen, 1975; Lubliner, 1990), are considered in the present study. For structures of such materials subjected to cyclic (mechanical and/or kinematical) loads, a steady cycle will be shown to exist and its basic characters pointed out. For this purpose the classical infinitesimal displacement theory will be applied and the material data will be treated as independent of temperature variations. Under these simplifying hypotheses, the equation set governing the steady-state response will be established together with the unique features of the relevant solution. The sensitivity of the steady cycle to the load parameters will be investigated and a number of results previously obtained by Polizzotto *et al.* (1990) and Polizzotto (1993a) for perfect plasticity will be extended to nonlinear hardening.

The plan of the paper is as follows. Section 2 is devoted to the description of the material model, Section 3 treats the steady-state response, Section 4 the equation set governing the steady cycle, Section 5 provides a classification of the steady cycles, and finally Sections 6 and 7 are devoted to the sensitivity analysis of the steady cycle. Section 8 considers the elastic region and conclusions are drawn in Section 9.

NOTATION—A compact notation is used throughout, with vectors and tensors denoted by bold face symbols. A scalar product is denoted by a dot for vectors and by a colon for second order tensors, e.g. $\mathbf{f}\cdot\mathbf{u} = f_i u_i$, $\boldsymbol{\sigma}:\boldsymbol{\varepsilon} = \sigma_{ij}\varepsilon_{ij}$, where the index summation rule holds. The scalar products between vectors ($\mathbf{n}$), second-order ($\boldsymbol{\sigma}$) and fourth-order ($\mathbf{A}$) tensors obey the rules: $(\boldsymbol{\sigma}\cdot\mathbf{n})_i = \sigma_{ij}n_j$, $(\mathbf{A}:\boldsymbol{\sigma})_{ij} = A_{ijhk}\sigma_{hk}$, $\boldsymbol{\sigma}:\mathbf{A}:\boldsymbol{\sigma} = \sigma_{ij}A_{ijhk}\sigma_{hk}$. The symbol $:=$ denotes equality by definition. Other symbols will be defined where they first appear.

2. THE MATERIAL MODEL

A (generalized standard) elastic–plastic material is considered as described by the following constitutive equations:

$$\phi(\boldsymbol{\sigma},\boldsymbol{\chi}) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\phi(\boldsymbol{\sigma},\boldsymbol{\chi}) = 0 \tag{1}$$

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial\phi(\boldsymbol{\sigma},\boldsymbol{\chi})}{\partial\boldsymbol{\sigma}}, \quad -\dot{\boldsymbol{\xi}} = \dot{\lambda}\frac{\partial\phi(\boldsymbol{\sigma},\boldsymbol{\chi})}{\partial\boldsymbol{\chi}} \tag{2}$$

$$\boldsymbol{\chi} = \frac{\partial\Psi(\boldsymbol{\xi})}{\partial\boldsymbol{\xi}} \tag{3}$$

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p + \boldsymbol{\varepsilon}^\theta, \quad \boldsymbol{\varepsilon}^e = \mathbf{A}:\boldsymbol{\sigma}. \tag{4}$$

Here, $\boldsymbol{\sigma}$ is the stress tensor, whereas $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^e$ and $\boldsymbol{\varepsilon}^p$ are tensors of total, elastic and plastic strains, and $\boldsymbol{\varepsilon}^\theta$ is the thermal strain tensor. $\mathbf{A}$ is the compliance fourth-order tensor of linear elasticity (with its usual symmetry and sign definiteness properties), $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ are dual internal variables (scalars, vectors or tensors, but here treated as tensors) mutually related by a convex thermodynamic potential, $\Psi(\boldsymbol{\xi})$, and referred to, respectively, as the stress-like and strain-like internal variables in the following. $\phi(\boldsymbol{\sigma}, \boldsymbol{\chi})$ is the yield function, by hypothesis convex and smooth in the $(\boldsymbol{\sigma}, \boldsymbol{\chi})$-space, also playing the role of plastic potential (associated plasticity).

As a consequence of the convexity of $\phi(\boldsymbol{\sigma}, \boldsymbol{\chi})$, the following inequality holds (see Halphen and Nguyen, 1975; Halphen, 1979; Polizzotto *et al.*, 1991):

$$(\boldsymbol{\sigma}-\bar{\boldsymbol{\sigma}}):\dot{\boldsymbol{\varepsilon}}^p - (\boldsymbol{\chi}-\bar{\boldsymbol{\chi}}):\dot{\boldsymbol{\xi}} \geqslant 0, \tag{5}$$

where the pairs $(\boldsymbol{\sigma}, \boldsymbol{\chi})$ and $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$ are mutually related by eqns (1) and (2), and the stress pair $(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}})$ is arbitrary but plastically admissible, i.e. $\phi(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{\chi}}) \leqslant 0$. Inequality (5) can also be viewed as a consequence of the *maximum intrinsic dissipation theorem*, that is

$$D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}}) = \max_{(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}})} (\bar{\boldsymbol{\sigma}}:\dot{\boldsymbol{\varepsilon}}^p - \bar{\boldsymbol{\chi}}:\dot{\boldsymbol{\xi}}) \quad \text{s.t. } \phi(\bar{\boldsymbol{\sigma}},\bar{\boldsymbol{\chi}}) \leqslant 0, \tag{6}$$

where $D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$ is the *intrinsic dissipation function* and "s.t." stands for "subject to". $D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$, as given by eqn (6), can be represented as

$$D(\dot{\varepsilon}^p, \dot{\xi}) = D_p - D_d = \sigma : \dot{\varepsilon}^p - \chi : \dot{\xi} \geq 0, \tag{7}$$

where $D_p = \sigma : \dot{\varepsilon}^p$ and $D_d = \chi : \dot{\xi}$ denote the *plastic* and *internal dissipations*, respectively, and the non-negativity of D is a consequence of the second thermodynamics principle.

By eqn (3), the internal dissipation rate density can also be written as

$$D_d = \chi : \dot{\xi} = \frac{\partial \Psi}{\partial \xi} : \dot{\xi} = \frac{\mathrm{d}}{\mathrm{d}t} \Psi(\xi). \tag{8}$$

Thus, on integration over the time interval $(0, t)$ from the virgin state ($\xi = \chi = \mathbf{0}$, $\Psi(\mathbf{0}) = 0$) to a state characterized by the internal variables ξ, the internal dissipation work spent to promote the change of the microstructure state within the unit volume, $w_d(t)$, is

$$w_d(t) = \int_0^t D_d \, \mathrm{d}\bar{t} = \Psi(\xi). \tag{9}$$

Making use of eqn (3) and on setting $\chi(\xi) = \partial \Psi / \partial \xi$, one can write $\phi(\sigma, \chi) = \phi(\sigma, \chi(\xi)) = \tilde{\phi}(\sigma, \xi)$ such that, by differentiation, one obtains

$$\frac{\partial \tilde{\phi}}{\partial \sigma} = \frac{\partial \phi}{\partial \sigma}, \quad \frac{\partial \tilde{\phi}}{\partial \xi} = \frac{\partial \chi}{\partial \xi} : \frac{\partial \phi}{\partial \chi} = \frac{\partial^2 \Psi}{\partial \xi \partial \xi} : \frac{\partial \phi}{\partial \chi}. \tag{10}$$

The latter relations, through multiplication by $\dot{\lambda}$ and by virtue of eqns (1)–(3), enable one to rewrite constitutive equations like eqns (1)–(3), but with $\tilde{\phi}$ in place of ϕ. Then, since $\tilde{\phi}(\sigma, \xi)$ is convex (and smooth) in the (σ, ξ)-space, the inequality

$$(\sigma - \bar{\sigma}) : \dot{\varepsilon}^p - (\xi - \bar{\xi}) : \dot{\chi} \geq 0 \tag{11}$$

holds for arbitrary pairs (σ, ξ) and $(\dot{\varepsilon}^p, \dot{\chi})$ corresponding to each other through the constitutive equations and for arbitrary plastically admissible pairs $(\bar{\sigma}, \bar{\xi})$, i.e. $\tilde{\phi}(\bar{\sigma}, \bar{\xi}) \leq 0$. Summing eqns (5) and (11) then yields the inequality

$$(\sigma - \bar{\sigma}) : \dot{\varepsilon}^p - \tfrac{1}{2}(\chi - \bar{\chi}) : \dot{\xi} - \tfrac{1}{2}(\xi - \bar{\xi}) : \dot{\chi} \leq 0, \tag{12}$$

which, rewritten in the form

$$(\sigma - \bar{\sigma}) : \dot{\varepsilon}^p - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} [(\chi - \bar{\chi}) : (\xi - \bar{\xi})] \geq 0, \tag{13}$$

is substantially equivalent to the one given by Halphen and Nguyen (1975). Inequalities (12) and (13) hold for any set $(\sigma, \chi, \xi, \dot{\varepsilon}^p, \dot{\chi}, \dot{\xi})$ complying with the constitutive equations (1)–(3) and for any plastically admissible set $(\bar{\sigma}, \bar{\chi}, \bar{\xi})$, i.e. $\phi(\bar{\sigma}, \chi(\bar{\xi})) \leq 0$. For a convex smooth yield function ϕ, the equality sign holds in eqns (12) and (13) if, and only if, either $\dot{\varepsilon}^p = \mathbf{0}$ and $\dot{\xi} = \dot{\chi} = \mathbf{0}$ (in which case σ, χ and ξ may be different from $\bar{\sigma}$, $\bar{\chi}$ and $\bar{\xi}$, respectively), or $\sigma = \bar{\sigma}$, $\chi = \bar{\chi}$, $\xi = \bar{\xi}$ (in which case $\dot{\varepsilon}^p$, $\dot{\xi}$ and $\dot{\chi}$ may be different from zero).

Following classical reasoning (see e.g. Martin, 1975), eqn (13) can be generalized by considering two states of the material and writing

$$(\sigma' - \sigma'') : (\dot{\varepsilon}^{p\prime} - \dot{\varepsilon}^{p\prime\prime}) - \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} [(\chi' - \chi'') : (\xi' - \xi'')] \geq 0, \tag{14}$$

where the primed and doubly primed quantities describe the two material states, respectively, both complying with eqns (1)–(3). It is worth noting that the equality sign in eqn (14) holds if, and only if, either $\dot{\varepsilon}^{p\prime} = \dot{\varepsilon}^{p\prime\prime} = \mathbf{0}$, $\dot{\xi}' = \dot{\xi}'' = \mathbf{0}$ and $\dot{\chi}' = \dot{\chi}'' = \mathbf{0}$ (in which case

$\boldsymbol{\sigma}'$, $\boldsymbol{\chi}'$ and $\boldsymbol{\xi}'$ may be different from $\boldsymbol{\sigma}''$, $\boldsymbol{\chi}''$ and $\boldsymbol{\xi}''$, respectively), or $\dot{\boldsymbol{\varepsilon}}^{p\prime} = \dot{\boldsymbol{\varepsilon}}^{p\prime\prime}$, $\dot{\boldsymbol{\xi}}' = \dot{\boldsymbol{\xi}}''$, $\dot{\boldsymbol{\chi}}' = \dot{\boldsymbol{\chi}}''$ and are not vanishing (in which case $\boldsymbol{\sigma}' = \boldsymbol{\sigma}''$, $\boldsymbol{\xi}' = \boldsymbol{\xi}''$, $\boldsymbol{\chi}' = \boldsymbol{\chi}''$).

The following specializations of the above material model are of interest for the purposes of the present paper (see Martin, 1975; Lemaitre and Chaboche, 1985; Lubliner, 1990).

(a) *Isotropic hardening material.* There is a single pair of internal scalar variables, say ξ_0 and χ_0, and the yield function has the form $\phi \equiv f(\boldsymbol{\sigma}) - k - \chi_0(\xi_0)$, where k is a positive constant and $\chi_0 = \mathrm{d}\Psi(\xi_0)/\mathrm{d}\xi_0$. Since $\dot{\xi}_0 = \dot{\lambda}$, the internal variable ξ_0 increases monotonically; since $\mathrm{d}\chi_0/\mathrm{d}\xi_0 > 0$, the yield surface expands homothetically during any plastic straining process.

(b) *Kinematically hardening material.* The yield function is of the form $\phi \equiv f(\boldsymbol{\sigma} - \boldsymbol{\chi}) - k$ such that $\dot{\boldsymbol{\xi}} = \dot{\boldsymbol{\varepsilon}}^p$, whereas the potential $\Psi(\boldsymbol{\xi})$ is in general chosen as a positive definite quadratic function. The yield surface translates during any plastic straining process while retaining its size and shape.

(c) *Hardening material with bounding surface.* The material hardens with an assigned rule as long as the yield surface is inside the bounding surface $F(\boldsymbol{\sigma}) = 0$, and behaves as a perfectly plastic material if the yield surface touches the bounding surface (Mróz, 1969; Krieg, 1975; Dafalias and Popov, 1975). A material model with hardening saturation capability belongs to the class of materials with bounding surface. Such a material hardens with a specific law as long as the internal dissipation work is less than some limit, e.g. $\Psi(\boldsymbol{\xi}) < \gamma$. At saturation, i.e. for $\Psi(\boldsymbol{\xi}) = \gamma$, provided that no unloading occurs, $\boldsymbol{\xi}$ can change but satisfies $\Psi(\boldsymbol{\xi}) = \gamma$, while the yield surface envelopes a surface $F(\boldsymbol{\sigma}) = 0$.

3. STEADY-STATE RESPONSE TO CYCLIC LOADS

A continuous body with an elastic–plastic material as described in the previous section is considered here. The undeformed body, referred to a Cartesian orthogonal coordinate system $\mathbf{x} = (x_1, x_2, x_3)$, occupies a region V of the three-dimensional Euclidean space and is restrained upon the part $\partial_u V$ of its boundary surface ∂V. It is loaded by external actions as body forces in V, tractions on $\partial_t V = \partial V / \partial_u V$, thermal strains in V and imposed displacements on $\partial_u V$. All these actions vary in a quasi-static manner with time $t \geqslant 0$ and are periodic with the period Δt. These same actions are in some way represented by the corresponding elastic response of the body, i.e. the stresses and displacements which arise in the body being treated as indefinitely elastic, and are henceforth denoted as $(\cdot)^E$. Obviously these stresses and displacements are periodic like the loads. The actual response of the body to the given loads and initial conditions (i.e. initial plastic strains and consequent stresses) is denoted by symbols such as $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{u}$, etc.

First, an intermediate result, useful for the subsequent developments, is derived. For this purpose, a hypothetical structural situation is considered in which the body suffers two distinct state evolutions, say $\boldsymbol{\sigma}'$, $\boldsymbol{\varepsilon}'$, ... and $\boldsymbol{\sigma}''$, $\boldsymbol{\varepsilon}''$, ... such that the difference fields $\Delta\boldsymbol{\sigma} = \boldsymbol{\sigma}' - \boldsymbol{\sigma}''$, etc., satisfy the identity

$$\Delta\boldsymbol{\sigma} : \Delta\dot{\boldsymbol{\varepsilon}}^p - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\boldsymbol{\chi} : \Delta\boldsymbol{\xi}) = 0 \text{ in } V, \quad \text{all } t \in J, \tag{15}$$

where J is some time interval, i.e. $J := \{t : t_0 \leqslant t \leqslant t_1\}$, with $t_0 \geqslant 0$. Considering what has been stated with regard to eqn (14) taken as an equality, eqn (15) implies the following:

$\Delta\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$, $\Delta\dot{\boldsymbol{\chi}} = \Delta\dot{\boldsymbol{\xi}} = \mathbf{0}$ in $V \times J$;

$\Delta\boldsymbol{\sigma} = \mathbf{0}$, $\Delta\boldsymbol{\chi} = \Delta\boldsymbol{\xi} = 0$ at points $\mathbf{x} \in V_p$ and times $t = t_p(\mathbf{x}) \in J_p$, where V_p and J_p collect points and times where plastic yielding occurs;

$\Delta\boldsymbol{\sigma}$, $\Delta\boldsymbol{\chi}$ and $\Delta\boldsymbol{\xi}$ may not vanish in V_p and for times $t \in J_e := J/J_p$ as well as in $V_e := V/V_p$ and for all $t \in J$.

Some additional consequences to eqn (15) can also be derived. First, $\Delta\boldsymbol{\chi}$ and $\Delta\boldsymbol{\xi}$ being time-independent in $V \times J$, it follows that $\Delta\boldsymbol{\chi} = \Delta\boldsymbol{\xi} = \mathbf{0}$ in V_p for all $t \in J$ (because these fields vanish there at certain times). Second, the residual stress rate difference field, $\Delta\dot{\boldsymbol{\sigma}}^R$, must vanish in V and for all $t \in J$ like $\Delta\dot{\boldsymbol{\varepsilon}}^p$ (to which it is the elastic response) and thus $\Delta\boldsymbol{\sigma}^R$ is time-independent in $V \times J$. Therefore, it can be stated that, upon validity of eqn (15), the following evolution uniqueness requisite is satisfied.

EVOLUTION UNIQUENESS REQUISITE—For an elastic–plastic structure for which two different state evolutions and some time interval J may be envisaged such as to satisfy eqn (15), the following properties hold true:

(i) The plastic strain rate difference field vanishes identically in V, i.e. $\Delta\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$ in V for all $t \in J$.

(ii) The internal variable difference fields, $\Delta\boldsymbol{\chi}$ and $\Delta\boldsymbol{\xi}$, are time-independent in $V \times J$ and in particular they identically vanish in the region V_p of plastic yielding, i.e. $\Delta\boldsymbol{\chi} = \mathbf{0}$, $\Delta\boldsymbol{\xi} = \mathbf{0}$ in V_p for all $t \in J$.

(iii) The residual stress difference field, $\Delta\boldsymbol{\sigma}^R$, is time-independent in $V \times J$.

(iv) The stress difference field, $\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}^E + \Delta\boldsymbol{\sigma}^R$, vanishes at points $\mathbf{x} \in V_p$ and at times $t = t_p(\mathbf{x}) \in J_p$.

In the special case in which the two state evolutions occur under the same load history, such that $\Delta\boldsymbol{\sigma}^E \equiv \mathbf{0}$, we have $\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}^R$ identically, properties (i) and (ii) above remain unaltered and (iii) and (iv) unify into the following point (v).

(v) The stress difference fields $\Delta\boldsymbol{\sigma}$ and $\Delta\boldsymbol{\sigma}^R$ identify with each other, are time-independent in $V \times J$ and in particular they vanish in $V_p \times J$.

With the above result in mind, let the body introduced at the beginning of this section be considered again, together with its actual response. On denoting by $\Delta(\cdot)$ the difference of values of a variable $(\cdot)$ at the instants t and $t+\Delta t$, for instance $\Delta\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}(\mathbf{x}, t+\Delta t) - \boldsymbol{\sigma}(\mathbf{x}, t)$, by virtue of eqns (1)–(4), which are continuously satisfied at all $t \geqslant 0$ and all $\mathbf{x} \in V$, eqn (14) applies and gives

$$l := \Delta\boldsymbol{\sigma} : \Delta\dot{\boldsymbol{\varepsilon}}^p - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta\boldsymbol{\chi} : \Delta\boldsymbol{\xi}) \geqslant 0 \text{ in } V, \quad \text{all } t \geqslant 0. \tag{16}$$

From the equality

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{A} : \dot{\boldsymbol{\sigma}} + \dot{\boldsymbol{\varepsilon}}^p + \dot{\boldsymbol{\varepsilon}}^\theta \tag{17}$$

written at times t and $t+\Delta t$, in view of the periodicity of $\boldsymbol{\varepsilon}^\theta(\mathbf{x}, t)$, it follows that

$$\Delta\dot{\boldsymbol{\varepsilon}}^p = \Delta\dot{\boldsymbol{\varepsilon}} - \mathbf{A} : \Delta\dot{\boldsymbol{\sigma}} \text{ in } V \tag{18}$$

for all $t \geqslant 0$ and then, on substitution of the latter equation into eqn (16) and with subsequent integration over the volume V, one has

$$\int_V l\,\mathrm{d}V = \int_V \Delta\boldsymbol{\sigma} : \Delta\dot{\boldsymbol{\varepsilon}}\,\mathrm{d}V - \int_V \Delta\boldsymbol{\sigma} : \mathbf{A} : \Delta\dot{\boldsymbol{\sigma}}\,\mathrm{d}V - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \Delta\boldsymbol{\chi} : \Delta\boldsymbol{\xi}\,\mathrm{d}V. \tag{19}$$

The first integral on the r.h.s. of eqn (19) vanishes because $\Delta\boldsymbol{\sigma}$, the difference between stresses in equilibrium with equal loads, is a self-stress field, whereas $\Delta\dot{\boldsymbol{\varepsilon}}$, the difference between two strain rate fields compatible with the same displacement rates on $\partial_u V$, is itself compatible with zero displacement rates on $\partial_u V$. Thus, eqn (19), by virtue of (16), can be written as

$$-\int_V l\,\mathrm{d}V = \frac{\mathrm{d}L}{\mathrm{d}t} \leqslant 0, \tag{20}$$

where L is the positive definite functional:

$$L := \frac{1}{2}\int_V \Delta\boldsymbol{\sigma} : \mathbf{A} : \Delta\boldsymbol{\sigma}\, dV + \frac{1}{2}\int_V \Delta\boldsymbol{\chi} : \Delta\boldsymbol{\xi}\, dV. \tag{21}$$

It thus results that, during the loading process, the state function L decreases monotonically with time. Since L cannot take negative values, it follows that L must stop decreasing at a certain time t_s, after which $dL/dt = 0$ at all times, hence $l = 0$ in V and for all $t \geqslant t_s$. This means that, for $t \geqslant t_s$, eqn (16) is satisfied as an equality and that the body experiences, under the same load history, two state evolutions satisfying eqn (15), with $t_0 = t_s$ and $t_1 = +\infty$. Therefore, the evolution uniqueness requisite applies here, with its properties (i), (ii) and (v). Additionally, since for the specific features of the problem at hand we have $\Delta\boldsymbol{\chi} = \Delta\boldsymbol{\xi} = \mathbf{0}$, $\Delta\boldsymbol{\varepsilon}^p = \mathbf{0}$ in the region $V_e = V/V_p$ where no plastic yielding occurs at times $t \geqslant t_s$, it follows that $\Delta\boldsymbol{\sigma}$ vanishes in V for all $t \geqslant t_s$. Therefore, one can state the following proposition.

Proposition 1. In a structure with a generalized standard elastic–plastic material and subjected to cyclic loads, there exists a stabilization time after which the (stabilized) response is characterized by stresses $\boldsymbol{\sigma}$, plastic strain rates $\dot{\boldsymbol{\varepsilon}}^p$ and internal variables $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$, all periodic with the same period of the loads.

The above result, derived under the hypothesis of smooth yield function, can also be considered valid for nonsmooth yield functions on the basis of arguments as in Martin (1975) for perfect plasticity.

Proposition 1 implies that, for loads (P^0, β) under which $V_p \neq \varnothing$, the body's stress and hardening states at any time $t \geqslant t_s$ are recovered after a complete cycle of time length Δt, but that this is not the case for the body's plastic strain state, as in fact the two plastic strain states generally differ from each other by a compatible strain field $\Delta\boldsymbol{\varepsilon}^p$, except when $\Delta\boldsymbol{\varepsilon}^p \equiv \mathbf{0}$ (plastic shakedown).

The long term (stabilized) response is, to some degree, dependent on the initial conditions. It is of interest to establish the essential characteristics of the long term response that are independent of the initial conditions. A reasoning path similar to that preceding Proposition 1 is useful for this purpose.

Let primes and double primes label two different responses of the body to the same loads, but with different initial conditions, and let the symbol $\Delta(\cdot)$ denote the difference between state variable values in the two responses at the same time, for instance $\Delta\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}'(\mathbf{x}, t) - \boldsymbol{\sigma}''(\mathbf{x}, t)$. Obviously, whatever the initial conditions, eqn (16) still holds, but with the difference fields having the new established meanings. Since eqns (18) and (19) can again be employed, it follows that the same formal procedure from eqns (16)–(21) applies, with the consequence that there exists some $t_s \geqslant 0$ after which $l = 0$. That is, the body experiences, under the same load history, two state evolutions for which eqn (15) is satisfied with $t_0 = t_s$ and $t_1 = +\infty$, and thus the evolution uniqueness requisite again applies with its properties (i), (ii) and (v). Since it can be excluded that $\Delta\boldsymbol{\varepsilon}^p$ may constitute a compatible strain field in V, one can state that $\Delta\boldsymbol{\varepsilon}^p = \mathbf{0}$ in V_p for $t \geqslant t_s$ and that in V_e some nonvanishing $\Delta\boldsymbol{\varepsilon}^p$, $\Delta\boldsymbol{\chi}$, $\Delta\boldsymbol{\xi}$ may exist as a consequence of different plastic straining processes and different initial conditions. Therefore, for $t \geqslant t_s$, it is found that $\dot{\boldsymbol{\varepsilon}}^{p\prime} = \dot{\boldsymbol{\varepsilon}}^{p\prime\prime}$, $\dot{\boldsymbol{\chi}}' = \dot{\boldsymbol{\chi}}''$, $\dot{\boldsymbol{\xi}}' = \dot{\boldsymbol{\xi}}''$ in the whole V and $\boldsymbol{\sigma}' = \boldsymbol{\sigma}''$, $\boldsymbol{\varepsilon}^{p\prime} = \boldsymbol{\varepsilon}^{p\prime\prime}$, $\boldsymbol{\chi}' = \boldsymbol{\chi}''$, $\boldsymbol{\xi}' = \boldsymbol{\xi}''$ in V_p (but not necessarily in V_e). In conclusion, it is possible to state the following proposition.

Proposition 2. In a structure with a generalized standard elastic–plastic material subjected to cyclic loads (but unspecified initial conditions), the long term response exhibits uniqueness for the plastic strain rates $\dot{\boldsymbol{\varepsilon}}^p$ and the internal variable rates $\dot{\boldsymbol{\chi}}$, $\dot{\boldsymbol{\xi}}$ in the whole V, as well as for the stresses $\boldsymbol{\sigma}$, plastic strains $\boldsymbol{\varepsilon}^p$ and internal variables $\boldsymbol{\chi}$, $\boldsymbol{\xi}$ within the region V_p of plastic yielding, whereas in the elastic region $V_e = V/V_p$, some time-independent $\boldsymbol{\varepsilon}^p$, $\boldsymbol{\chi}$, $\boldsymbol{\xi}$ may exist with values dependent on the initial conditions and on the transient straining process as well.

The long term stabilized response to given cyclic loads, but with unspecified initial conditions, is usually referred to as the *steady-state response*, or the *steady-cycle response*, or simply the *steady cycle*.

4. EQUATIONS GOVERNING THE STEADY CYCLE

The steady cycle of a structure for given cyclic loads can be determined directly, without taking into consideration the transient response phase. This goal can be achieved by solving an *ad hoc* equation set worth being specified. In the light of the results of Section 3, these equations read:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^R, \quad \boldsymbol{\chi} = \frac{\partial \Psi(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}} \text{ in } V \times (0, \Delta t) \tag{22}$$

$$\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda}\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0 \text{ in } V \times (0, \Delta t) \tag{23}$$

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial \phi}{\partial \boldsymbol{\sigma}}, \quad -\dot{\boldsymbol{\xi}} = \dot{\lambda}\frac{\partial \phi}{\partial \boldsymbol{\chi}} \text{ in } V \times (0, \Delta t) \tag{24}$$

$$C^T \boldsymbol{\sigma}^R = \mathbf{0} \text{ in } V \times (0, \Delta t), \qquad \boldsymbol{\sigma}^R \cdot \mathbf{n} = \mathbf{0} \text{ on } \partial_t V \times (0, \Delta t) \tag{25a}$$

$$C\dot{\mathbf{u}}^R = \mathbf{A} : \dot{\boldsymbol{\sigma}}^R + \dot{\boldsymbol{\varepsilon}}^p \text{ in } V \times (0, \Delta t), \quad \dot{\mathbf{u}}^R = \mathbf{0} \text{ on } \partial_u V \times (0, \Delta t) \tag{25b}$$

$$\Delta \boldsymbol{\varepsilon}^p := \int_0^{\Delta t} \dot{\boldsymbol{\varepsilon}}^p \, \mathrm{d}t = C\mathbf{v} \text{ in } V, \quad \mathbf{v} = \mathbf{0} \text{ on } \partial_u V \tag{26}$$

$$\Delta \boldsymbol{\xi} := \int_0^{\Delta t} \dot{\boldsymbol{\xi}} \, \mathrm{d}t = \mathbf{0} \text{ in } V. \tag{27}$$

Here, $\boldsymbol{\sigma}^E = \boldsymbol{\sigma}^E(\mathbf{x}, t)$, $0 \leqslant t \leqslant \Delta t$, is the elastic stress response (assumed known) and the stress field $\boldsymbol{\sigma}$ is expressed as the superposition of $\boldsymbol{\sigma}^E$ with the residual stress field $\boldsymbol{\sigma}^R$, the latter being associated with the residual displacement field $\mathbf{u}^R$; furthermore $\mathbf{n}$ is the unit external normal to ∂V, C denotes the compatibility differential operator and C^T its adjoint, i.e.

$$C(\cdot) = \tfrac{1}{2}[\text{grad}(\cdot) + \text{grad}(\cdot)^T], \quad C^T(:) = \text{div}(:).$$

Thus, eqn (25a) qualifies $\boldsymbol{\sigma}^R$ as a self-stress field, whereas eqn (25b) states that $\dot{\boldsymbol{\sigma}}^R$ and $\dot{\mathbf{u}}^R$ are stress rates and displacement rates elastically associated with $\dot{\boldsymbol{\varepsilon}}^p$. Additionally, eqn (26) states that the ratchet strains $\Delta\boldsymbol{\varepsilon}^p$ (i.e. the net plastic strains accumulated in the complete cycle) are compatible with the displacements $\mathbf{v}$ vanishing on $\partial_u V$, such that the residual stresses $\boldsymbol{\sigma}^R$ existing in V at $t = 0$ are reconstituted at $t = \Delta t$; analogously, eqn (27) requires that the ratchet strain-like internal variables, $\Delta\boldsymbol{\xi}$, vanish everywhere in V, such that the hardening state at $t = 0$ is reconstituted at $t = \Delta t$ in the whole V. The above properties, as summarized by eqns (25)–(27), will be referred to by saying that the strain rate history pair ($\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\boldsymbol{\xi}}$) within (0, Δt) constitutes a *Plastic Accumulation Mechanism* (PAM) characterized by the ratchet strains $\Delta\boldsymbol{\varepsilon}^p$ with their related displacements $\mathbf{v} = \Delta\mathbf{u}^R$ (see Polizzotto *et al.*, 1991).

Equations (22)–(27) resemble an evolutive problem of elastoplasticity in the framework of internal variable formulations, but with the customary initial conditions replaced by suitable time-integral conditions, i.e. eqns (26) and (27). The above equation set is generally very difficult to solve and simplified analysis methods are needed in practice.

On assigning to a variable $(\cdot)$ values at times $T = t + n\Delta t$ ($n = 1, 2, \ldots$), with the rule $(\cdot)|_{t+n\Delta t} = (\cdot)|_t$ for all $t \in (0, \Delta t)$, any state or evolution variable pertaining to the/a solution to eqns (22)–(27) can be envisaged as a function of $T \geqslant 0$. It can be easily shown that the variables $\boldsymbol{\sigma}$, $\dot{\boldsymbol{\varepsilon}}^p$ and $\boldsymbol{\chi}$ as functions of T are periodic with period Δt, just like $\boldsymbol{\xi}$, whose periodicity is directly stated by (27). In fact, the periodity of $\boldsymbol{\sigma}^R$ (hence of $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^R$) stems from the fact that the plastic strains accumulated from $t = 0$ to $t = \Delta t$ are compatible and thus the corresponding stresses, $\boldsymbol{\sigma}^R$, at $t = 0$ coincide with those at $t = \Delta t$. The periodicity of $\boldsymbol{\chi}$ is a direct consequence of the periodicity of $\boldsymbol{\xi}$ through the second eqn (22). Finally, the periodicity of $\dot{\boldsymbol{\varepsilon}}^p$ stems from the periodicity of $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ and $\dot{\lambda}$ through the first eqn (24), with

$\dot{\lambda}$ uniquely determined in terms of $\dot{\sigma}$, σ, χ and ξ. Namely, from the plastic consistency condition $\dot{\phi} = 0$, one obtains (Martin, 1975; Lubliner, 1990):

$$\dot{\lambda} = \frac{1}{H}\left\langle \frac{\partial \phi}{\partial \sigma} : \dot{\sigma} \right\rangle, \tag{28}$$

where $H = H(\sigma, \xi)$ is the hardening modulus, i.e.

$$H = \frac{\partial \phi}{\partial \chi} : \frac{\partial^2 \Psi}{\partial \xi \partial \xi} : \frac{\partial \phi}{\partial \chi} > 0 \tag{29}$$

and $\langle \cdot \rangle$ is the Macauley operator, namely $\langle x \rangle = x$ for $x \geqslant 0$ and $\langle x \rangle = 0$ for $x < 0$. By eqn (25b), $\dot{\mathbf{u}}^R$ turns out to be periodic too.

It can now be proved that the solution to eqns (22)–(27) is unique for all, except that σ^R, χ and ξ are uniquely determined only in the region $V_p \subseteq V$ (if any) where plastic yielding takes place. For this purpose, let there exist, by absurdity, two solutions respectively labeled by primes and double primes, and let the symbol $\Delta(\cdot)$ denote the difference of values of the typical variable $(\cdot)$ in the two solutions, but at the same time, e.g. $\Delta\sigma(\mathbf{x}, t) = \sigma'(\mathbf{x}, t) - \sigma''(\mathbf{x}, t)$. Applying eqn (14) enables one to write an inequality formally equal to eqn (16). Since an equality formally identical to eqn (18) holds, it follows that eqns (19) and (20) still hold true, namely

$$\int_V l \, \mathrm{d}V = -\frac{\mathrm{d}L}{\mathrm{d}t} \geqslant 0 \text{ all } t \in (0, \Delta t), \tag{30}$$

where L is given by eqn (21), but with the difference fields having the new meanings. Since the latter fields take equal values at $t = 0$ and $t = \Delta t$, respectively, integration of eqn (30) over $(0, \Delta t)$ then yields

$$\int_0^{\Delta t} \int_V l \, \mathrm{d}V = L(0) - L(\Delta t) = 0 \tag{31}$$

and, as a consequence, l must vanish identically in $V \times (0, \Delta t)$. This means that the body suffers, under a single load history, two state evolutions such as to satisfy eqn (15) with $t_0 = 0$ and $t_1 = \Delta t$, and thus the evolution uniqueness requisite applies to the present case with its properties (i), (ii) and (v). Thus, one can state that: (i) $\Delta\dot{\varepsilon}^p$, $\Delta\dot{\chi}$ and $\Delta\dot{\xi}$ vanish in $V \times (0, \Delta t)$, then $\dot{\varepsilon}^p$, $\dot{\chi}$ and $\dot{\xi}$ are unique in $V \times (0, \Delta t)$ and (ii) $\Delta\sigma$, $\Delta\chi$ and $\Delta\xi$ vanish in $V_p \times (0, \Delta t)$, then σ, χ and ξ are unique in $V_p \times (0, \Delta t)$. The statement is so proved. Obviously, if V_p is empty, then $\dot{\varepsilon}^p$, $\dot{\chi}$ and $\dot{\xi}$ vanish identically, whereas σ^R, χ and ξ turn out to be indetermined time-independent fields in V (elastic shakedown).

The solution to eqns (22)–(27) provides the steady-cycle response of Section 3. In order to prove that, let primed symbols be used for the solution to eqns (22)–(27) and doubly primed symbols for the steady-state response. A piece of the steady-state response of time length Δt is extracted and a local time t, $0 \leqslant t \leqslant \Delta t$, is introduced such that the primed and doubly primed variables are all functions of t and are related with the same load at every $t \in (0, \Delta t)$. Since both solutions satisfy the constitutive equations, eqn (14) applies, enabling one to write an inequality formally identical to eqn (16). Since eqn (18) still holds for all $t \in (0, \Delta t)$, the reasoning from eqn (16) to eqn (21) can be followed here. Then, considering that an equality like eqn (31) can be shown to hold, with l and L as specified by eqns (16) and (21), one has that $l = 0$ identically. Again, the body experiences, under the same load history, two state evolutions satisfying eqn (15) with $t_0 = 0$ and $t_1 = \Delta t$, and the evolution uniqueness requisite holds true with its properties (i), (ii) and (v). Thus, one can state that, at every $t \in (0, \Delta t)$, $\dot{\varepsilon}^{p\prime} = \dot{\varepsilon}^{p\prime\prime}$, $\dot{\chi}' = \dot{\chi}''$ and $\dot{\xi}' = \dot{\xi}''$ everywhere in V and, furthermore, $\sigma' = \sigma''$, $\chi' = \chi''$, $\xi' = \xi''$ in V_p (where plastic yielding takes place), but σ', χ' and ξ' may not coincide with σ'', χ'' and ξ'' in the region $V_e = V/V_p$ (where σ', χ' and ξ' are not uniquely determined). This proves the above statement, such that the solution to eqns (22)–(27) can be referred to as the steady cycle. It also follows from the above that the steady cycle is independent of the load cycle origin.

The above circumstances suggest one to consider the residual stress tensor σ^R as the superposition of two stress tensors, namely

$$\sigma^R(\mathbf{x}, t) = \rho(\mathbf{x}) + \tau(\mathbf{x}, t), \quad 0 \leqslant t \leqslant \Delta t, \tag{32}$$

where $\rho(\mathbf{x})$ denotes [according to a terminology introduced by Polizzotto (1993a, c)] the *post-transient residual stresses*, i.e. the residual stresses associated with the plastic strains $\varepsilon_s^p(\mathbf{x})$ and the internal variables $\chi_s(\mathbf{x})$ and $\xi_s(\mathbf{x})$ at the stabilization time, whereas τ denotes the pure cyclic residual stresses, i.e. the residual stresses associated with the additional plastic strains arising in the steady cycle. This implies that, at some $t^* \in (0, \Delta t)$, $\sigma^R(\mathbf{x}, t^*) = \rho(\mathbf{x})$ and $\tau(\mathbf{x}, t^*) = \mathbf{0}$ in V. However, it is always possible, perhaps with the aid of a suitable choice of the load cycle origin and of the initial conditions, to take $t^* = 0$, i.e. to make the stabilization time occur at the beginning of some subsequent cycle. With this choice,

$$\tau(\mathbf{x}, t) = \int_0^t \dot{\tau}(\mathbf{x}, \bar{t})\, d\bar{t}, \tag{33}$$

and τ turns out to be uniquely determined in $V \times (0, \Delta t)$, just like $\dot{\tau} = \dot{\sigma}^R$, whereas ρ is uniquely determined only within V_p, just like σ^R. Finally, it is worth noting that the steady cycle is characterized by a vanishing cycle internal dissipation work density, i.e. $w_d = [\Psi(\xi)]_0^{\Delta t} = 0$. Thus, the cycle plastic dissipation work density, $w_p = \int_0^{\Delta t} D_p\, dt$, coincides with the cycle intrinsic dissipation work density, and hence is totally externally dispersed.

5. CATEGORIZATION OF THE STEADY CYCLE

The steady cycle in a given body, governed by eqns (22)–(27), can be categorized on the basis of the related PAM ($\dot{\varepsilon}^p$, $\dot{\xi}$), in perfect agreement with the case of a perfectly plastic material (Polizzotto, 1993a). The following three categories of steady cycles can thus be distinguished.

(1) *Elastic shakedown*. The PAM is a trivial one, namely $\dot{\varepsilon}^p$ and $\dot{\xi}$ vanish identically in $V \times (0, \Delta t)$. This means that the body responds to the loads in an elastic manner after the stabilization time and that no further plastic deformations are produced in addition to those occurring in the transient response phase (the latter being absent in the case of a fully elastic deformation process). Static and kinematic (elastic) shakedown theorems have already been established to discriminate shakedown from nonshakedown loads, and methods are available for specifying the shakedown limit loads (Halphen, 1979; Maier, 1987; Maier and Novati, 1987; Polizzotto *et al.*, 1991). For an isotropic hardening material, elastic shakedown is the only form of steady cycle allowed, as for such material one has $\Delta\xi_0 = \Delta\lambda = 0$ in V, hence $\dot{\lambda} = \dot{\xi}_0 = 0$ and $\dot{\varepsilon}^p = \mathbf{0}$ identically in $V \times (0, \Delta t)$, whatever the cyclic load. In the case of elastic shakedown, eqns (22)–(27) reduce drastically because $\dot{\lambda}$ vanishes identically, together with $\dot{\sigma}^R$, $\dot{\mathbf{u}}^R$ and $\dot{\chi}$, such that σ^R and χ turn out to be time-independent, and the only meaningful equation is the yield condition $\phi(\sigma, \chi) < 0$ in $V \times (0, \Delta t)$.

(2) *Plastic shakedown*. The PAM is characterized by identically vanishing ratchet plastic strains, i.e. $\Delta\varepsilon^p = \mathbf{0}$ in V, such that the plastic strains ε^p in the steady cycle are periodic. These strains generally remain small, but their alternating character produces low-cycle fatigue (*alternating plasticity collapse*). For kinematically hardening materials, plastic shakedown occurs under any cyclic load above the elastic shakedown limit load, as, with such a material, $\Delta\xi = \Delta\varepsilon^p = \mathbf{0}$ in V whatever the given load.

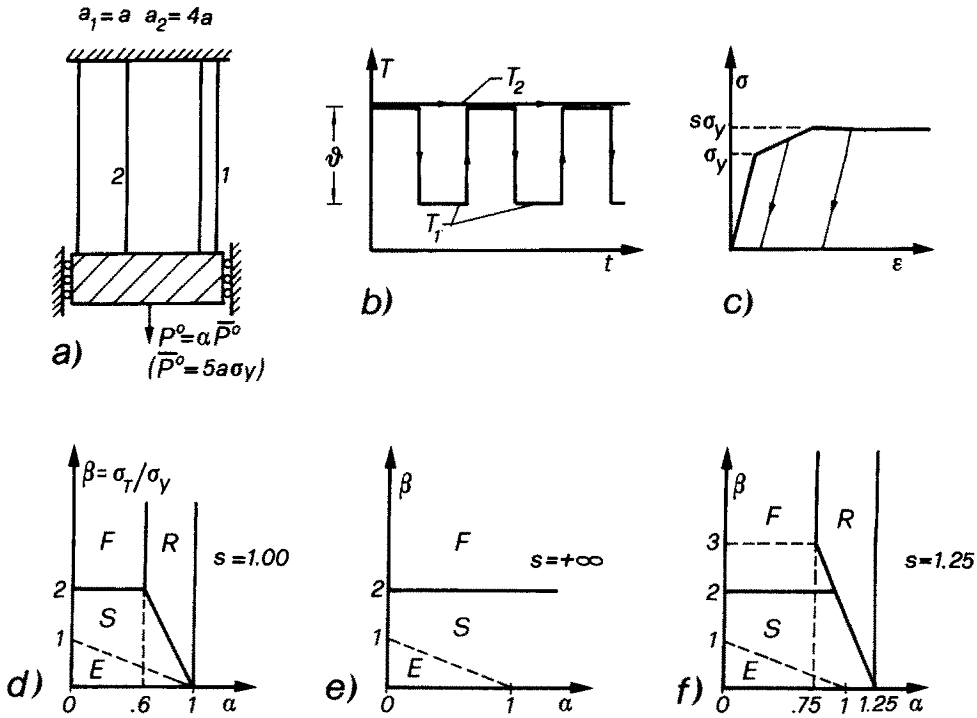

Fig. 1. Two-bar unsymmetric one-degree-of-freedom system with bar 1 subjected to temperature variation cycles and bar 2 taken at constant temperature. (a) Structural and geometrical sketch. (b) Temperature variation histories. (c) Stress–strain diagram (elastic–kinematically hardening–perfectly plastic material). (d)–(f) Bree diagrams for different values of the parameter s (σ_T = max thermoelastic stress).

(3) *Ratchetting.* The PAM is characterized by nonvanishing ratchet plastic strains in at least a portion of the body, i.e. $\Delta\boldsymbol{\varepsilon}^p \not\equiv \mathbf{0}$. Thus, the plastic strains increase by a constant amount at every cycle and soon become too large (*incremental collapse*). Ratchetting can occur with a perfectly plastic material, as well as with a hardening material with a bounding surface (or with hardening saturation).

The type of steady cycle that is actually established under a given load depends, for a given structure, on the load parameters. In the space of these parameters, the steady cycle can be mapped into a convex domain $B = B_S \cup B_F \cup B_R$, where B_S, B_F and B_R are the subdomains (or zones) of elastic shakedown (and purely elastic behavior as well), plastic shakedown and ratchetting, respectively. B is known as the *interaction diagram* (or generalized Bree diagram) in the related literature (see Ponter, 1983; Gokhfeld and Cherniavsky, 1980; Polizzotto, 1993a) (Fig. 1a–f). Methods for specifying the elastic shakedown boundary are already available (see Maier, 1987; Maier and Novati, 1987; Polizzotto *et al.*, 1991). Methods like those proposed by Ponter (1983), Ponter and Karadeniz (1985a, b), and Ponter *et al.* (1990) for cyclically hardening materials and thermal loading, and by Polizzotto (1993b) for perfect plasticity and general cyclic loading may be generalized to the present nonlinear hardening context, but this task is open to future research work.

For design purposes, criteria for predicting the zone to which the steady cycle belongs are of primary importance. The elastic shakedown theorems are available for applications to the B_S zone; the plastic shakedown theorems given by Polizzotto (1993b) for perfect plasticity may be extended to nonlinear hardening, but this will be the object of a future paper.

6. LOAD SENSITIVITY OF THE STEADY CYCLE

This section is devoted to investigating the sensitivity of the steady cycle to changes in the load parameters. Contributions to this topic were given by Ainsworth *et al.* (1980) with reference to the influence of the steady load on the elastic shakedown boundary. Here, a more general viewpoint is taken, similar to analogous studies for perfect plasticity (Polizzotto *et al.*, 1990; Polizzotto, 1993a). For this purpose, let the given load be synthetically represented as

$$P(t) = \beta \bar{P}^c(t) + P^0, \quad 0 \leqslant t \leqslant \Delta t, \tag{34}$$

where $\bar{P}^c$ is a specified periodic load, β a scalar parameter and P^0 a time-independent (or permanent) load. P^0 can in principle be an arbitrary permanent load below the ultimate plastic failure values, but on occasion it may belong to a narrower set. Obviously, the elastic response to the load of eqn (34) can be represented in an analogous way, i.e.

$$\boldsymbol{\sigma}^E(\mathbf{x}, t) = \beta \bar{\boldsymbol{\sigma}}^c(\mathbf{x}, t) + \boldsymbol{\sigma}^0(\mathbf{x}), \tag{35}$$

where $\bar{\boldsymbol{\sigma}}^c(\mathbf{x}, t)$ and $\boldsymbol{\sigma}^0(\mathbf{x})$ are the elastic stress responses to $\bar{P}^c(t)$ and P^0, respectively.

By integration of eqn (7) over $V \times (0, \Delta t)$ and considering that no contribution is given by the internal dissipation term, one can write the equality

$$W := \int_0^{\Delta t} \int_V D(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})\, \mathrm{d}V\, \mathrm{d}t = W_p, \tag{36}$$

i.e. the total intrinsic dissipation work, W, coincides with the total plastic work, W_p, the latter being expressed, by virtue of eqns (22) and (32), as

$$W_p = \int_0^{\Delta t} \int_V D_p\, \mathrm{d}V\, \mathrm{d}t = \int_0^{\Delta t} \int_V (\beta \bar{\boldsymbol{\sigma}}^c + \boldsymbol{\sigma}^0 + \boldsymbol{\rho} + \boldsymbol{\tau}) : \dot{\boldsymbol{\varepsilon}}^p\, \mathrm{d}V\, \mathrm{d}t = \beta \int_0^{\Delta t} \int_V \bar{\boldsymbol{\sigma}}^c : \dot{\boldsymbol{\varepsilon}}^p\, \mathrm{d}V\, \mathrm{d}t$$
$$+ \int_V \boldsymbol{\sigma}^0 : \Delta \boldsymbol{\varepsilon}^p\, \mathrm{d}V + \int_V \boldsymbol{\rho} : \Delta \boldsymbol{\varepsilon}^p\, \mathrm{d}V + \int_0^{\Delta t} \int_V \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^p\, \mathrm{d}V\, \mathrm{d}t. \tag{37}$$

The third integral on the r.h.s. of eqn (37) vanishes by the virtual work principle ($\boldsymbol{\rho}$ is self-equilibrated with zero tractions on $\partial_t V$, $\Delta \boldsymbol{\varepsilon}^p$ is compatible with zero displacements on $\partial_u V$), whereas the last integral can be transformed as

$$\int_0^{\Delta t} \int_V \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^p\, \mathrm{d}V\, \mathrm{d}t = \int_0^{\Delta t} \int_V \boldsymbol{\tau} : (\dot{\boldsymbol{\varepsilon}}^R - \mathbf{A} : \dot{\boldsymbol{\tau}})\, \mathrm{d}V\, \mathrm{d}t$$
$$= \int_0^{\Delta t} \int_V \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^R\, \mathrm{d}V\, \mathrm{d}t - \frac{1}{2} \int_V \boldsymbol{\tau} : \mathbf{A} : \boldsymbol{\tau}\, \mathrm{d}V \Big|_0^{\Delta t} = 0, \tag{38}$$

i.e. it vanishes as a consequence of the vanishing of the last two integral terms of eqn (38)—namely, the one due to the virtual work principle and the other due to the periodicity of $\boldsymbol{\tau}$. Thus, the expression of W_p in eqn (37) simplifies and eqn (36) can be written as

$$W = \beta \bar{A}^c + A^0, \tag{39}$$

where

$$A^0 := \int_V \boldsymbol{\sigma}^0 : \Delta \boldsymbol{\varepsilon}^p\, \mathrm{d}V \quad \text{(primary accumulation parameter)} \tag{40a}$$

$$\bar{A}^c := \int_0^{\Delta t} \int_V \bar{\boldsymbol{\sigma}}^c : \dot{\boldsymbol{\varepsilon}}^p\, \mathrm{d}V\, \mathrm{d}t \quad \text{(secondary net accumulation parameter).} \tag{40b}$$

Notice that $A^0 = 0$ for any load P^0 representing a kinematical load (i.e. thermal load and/or imposed displacements upon $\partial_u V$), as well as for P^0, being a mechanical load in the case of plastic shakedown.

With these premises in mind, let δP^0 and $\delta\beta$ be arbitrary small load increments and let $(P^0+\delta P^0,\ \beta+\delta\beta)$ be loads in a small neighborhood of a given load $(P^0,\ \beta)$. On the assumptions that no instability phenomena are present and the steady cycle varies with continuity on changing the load, the varied steady cycle can be expressed as $\boldsymbol{\sigma}+\delta\boldsymbol{\sigma}$, $\mathbf{u}+\delta\mathbf{u}$, etc., where $\delta\boldsymbol{\sigma}$, $\delta\mathbf{u}$, etc. denote the variable increments corresponding to the load increments δP^0, $\delta\beta$.

As a consequence of eqn (14), one can write

$$\delta^2 D = \delta\boldsymbol{\sigma} : \delta\dot{\boldsymbol{\varepsilon}}^p - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\chi} : \delta\boldsymbol{\xi}) \geqslant 0 \text{ in } V\times(0,\Delta t), \tag{41}$$

to be satisfied by the relevant increments. On remarking that the integral

$$\int_0^{\Delta t}\int_V \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\chi} : \delta\boldsymbol{\xi})\,\mathrm{d}V\,\mathrm{d}t = \frac{1}{2}\int_V \delta\boldsymbol{\xi} : \frac{\partial^2\Psi}{\partial\boldsymbol{\xi}\partial\boldsymbol{\xi}} : \delta\boldsymbol{\xi}\,\mathrm{d}V\bigg|_0^{\Delta t} = 0, \tag{42}$$

i.e. it vanishes due to the periodicity of $\boldsymbol{\xi}$ and $\delta\boldsymbol{\xi}$, and in consideration that

$$\delta\boldsymbol{\sigma} = \delta\beta\bar{\boldsymbol{\sigma}}^c(\mathbf{x},t) + \delta\boldsymbol{\sigma}^0(\mathbf{x}) + \delta\boldsymbol{\rho}(\mathbf{x}) + \delta\boldsymbol{\tau}(\mathbf{x},t), \tag{43}$$

one obtains from eqn (41), after integration over $V\times(0,\ \Delta t)$:

$$\delta^2 W = \int_0^{\Delta t}\int_V \delta^2 D\,\mathrm{d}V\,\mathrm{d}t = \int_0^{\Delta t}\int_V \delta\boldsymbol{\sigma} : \delta\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}V\,\mathrm{d}t = \delta\beta\delta\bar{A}^c + \delta^2 A^0$$

$$+\int_V \delta\boldsymbol{\rho} : \Delta\delta\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}V + \int_0^{\Delta t}\int_V \delta\boldsymbol{\tau} : \delta\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}V\,\mathrm{d}t \geqslant 0, \tag{44}$$

where $\delta\bar{A}^c$ and $\delta^2 A^0$ are variations of $\bar{A}^c$ and A^0, namely

$$\delta^2 A^0 = \int_V \delta\boldsymbol{\sigma}^0 : \Delta\delta\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}V, \quad \delta\bar{A}^c = \int_0^{\Delta t}\int_V \bar{\boldsymbol{\sigma}}^c : \delta\dot{\boldsymbol{\varepsilon}}^p\,\mathrm{d}V\,\mathrm{d}t. \tag{45}$$

As the last two integrals on the r.h.s. of eqn (44) vanish by the virtual work principle ($\delta\boldsymbol{\rho}$ is a self-stress field, $\Delta\delta\dot{\boldsymbol{\varepsilon}}^p = \delta\Delta\dot{\boldsymbol{\varepsilon}}^p$ is a compatible field) and, respectively, by eqn (38) written for $\delta\boldsymbol{\tau}$ and $\delta\dot{\boldsymbol{\varepsilon}}^p$, then eqn (44) simplifies to

$$\delta^2 W = \delta\beta\delta\bar{A}^c + \delta^2 A^0 \geqslant 0, \tag{46}$$

an extension and generalization to the present context of an analogous inequality given for perfect plasticity in Polizzotto *et al.* (1990) and Polizzotto (1993a), where permanent loads of only mechanical type were considered.

The above inequality interprets the sensitivity of the steady cycle to small load increments by stating that in any small change of the load parameters from $(P^0,\ \beta)$ to $(P^0+\delta P^0,\ \beta+\delta\beta)$, the consequent (small) change in the steady cycle is characterized by a non-negative second variation of the total plastic (and also intrinsic) dissipation work in the cycle.

For loads $(P^0,\ \beta)\in B_S$, the steady cycle is always characterized by an identically vanishing PAM $(\dot{\boldsymbol{\varepsilon}}^p,\ \dot{\boldsymbol{\xi}})$, such that $W=0$ and $\delta^2 W = 0$ everywhere in B_S and for arbitrary small increments. Thus, one can say that in B_S the steady cycle is fully *insensitive* to small load increments.

In analogy with the perfect plasticity case, eqn (46) can also be interpreted as a *stability principle* for the steady cycle (see Polizzotto *et al.*, 1990; Polizzotto, 1993a), but this point is not pursued here for simplicity.

7. INSENSITIVITY TO LOAD INCREMENTS

Let a load $(P^0, \beta) \notin B_S$ be considered together with the corresponding steady cycle and let correspondingly, by hypothesis, $\delta^2 W = 0$ for a certain class of load increments $(\delta P^0, \delta\beta)$. As a consequence of this hypothesis, eqn (41) is satisfied as an identity, i.e.

$$\delta^2 D = \delta\boldsymbol{\sigma} : \delta\dot{\boldsymbol{\varepsilon}}^p - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{\chi} : \delta\boldsymbol{\xi}) = 0 \text{ in } V \times (0, \Delta t). \tag{47}$$

The latter identity is formally coincident with eqn (15), but the incremental fields $\delta(\cdot)$ replace the difference fields $\Delta(\cdot)$ and $t_0 = 0$ and $t_1 = \Delta t$. Thus, the evolution uniqueness requisite applies here in its general format, with its properties (i), (ii) and (iii), and enables one to state that $\delta\dot{\boldsymbol{\varepsilon}}^p = \mathbf{0}$, $\delta\dot{\boldsymbol{\chi}} = \mathbf{0}$, $\delta\dot{\boldsymbol{\xi}} = \mathbf{0}$ in $V \times (0, \Delta t)$, and that $\delta\boldsymbol{\chi}$ and $\delta\boldsymbol{\xi}$ are time-independent fields of V, vanishing in V_p. Also, since $\delta\dot{\boldsymbol{\sigma}}^R \equiv \delta\dot{\boldsymbol{\tau}} \equiv \mathbf{0}$, hence $\delta\boldsymbol{\tau} \equiv \mathbf{0}$, eqn (43) now reads

$$\delta\boldsymbol{\sigma} = \delta\beta\bar{\boldsymbol{\sigma}}^c(\mathrm{x}, t) + \delta\boldsymbol{\sigma}^0(\mathrm{x}) + \delta\boldsymbol{\rho}(\mathrm{x}) = \mathbf{0} \text{ in } V_p \times (0, \Delta t)_p. \tag{48}$$

The set $(0, \Delta t)_p$ (collecting the times at which plastic yielding occurs) may contain a single instant only in the case of elastic shakedown or even of plastic collapse, but both circumstances are to be excluded here. Therefore, eqn (48) can be satisfied if, and only if, $\delta\beta = 0$ such that it reduces to

$$\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^0 + \delta\boldsymbol{\rho} = 0 \text{ in } V_p \times (0, \Delta t). \tag{49}$$

It transpires from eqn (49) that the region V_p cannot contain the application points of the mechanical part of δP^0 since $\delta\boldsymbol{\sigma}^0$ must constitute a self-stress field in V_p.

Summarizing the above results, the following can be stated:

(a) At a load point $(P^0, \beta) \notin B_S$, the only load increments such that $\delta^2 W = 0$ are increments of the permanent load with $\delta\beta = 0$.

(b) The steady cycle pertaining to (P^0, β) is insensitive to increments of the permanent load for which $\delta^2 W = 0$ (if any), in the sense that, upon such a load increment, the related PAM $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$ remains unaltered, as do the state variables $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ in $V_p \times (0, \Delta t)$, and also the region V_p of plastic yielding.

(c) The load increment δP^0 for which $\delta^2 W = 0$ at (P^0, β), if any, must have the application points of its mechanical part within the elastic region $V_e = V / V_p$, where a stress change $\delta\boldsymbol{\sigma}$ arises such as to equilibrate δP^0.

Another consequence of eqn (47) can be derived from the vanishing integral

$$\int_0^{\Delta t}\int_V \delta\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \,\mathrm{d}V\,\mathrm{d}t = \int_V \delta\boldsymbol{\sigma}^0 : \Delta\boldsymbol{\varepsilon}^p \,\mathrm{d}V + \int_V \delta\boldsymbol{\rho} : \Delta\boldsymbol{\varepsilon}^p \,\mathrm{d}V = 0, \tag{50}$$

where $\dot{\boldsymbol{\varepsilon}}^p$ pertains to the steady cycle of the load (P^0, β). The considered integral vanishes because $\dot{\boldsymbol{\varepsilon}}^p \equiv \mathbf{0}$ in V_e (where $\delta\boldsymbol{\sigma} \not\equiv \mathbf{0}$), and $\delta\boldsymbol{\sigma} \equiv \mathbf{0}$ in V_p (where $\dot{\boldsymbol{\varepsilon}}^p \not\equiv \mathbf{0}$). Since the last integral on the r.h.s. of eqn (50) is zero ($\delta\boldsymbol{\rho}$ is self-equilibrated, $\Delta\boldsymbol{\varepsilon}^p$ is compatible), eqn (50) reduces to

$$\int_V \delta\boldsymbol{\sigma}^0 : \Delta\boldsymbol{\varepsilon}^p \, \mathrm{d}V = 0. \tag{51}$$

As the latter equation must hold for arbitrary choices of δP^0, and hence of $\delta\boldsymbol{\sigma}^0$ in V, it follows that $\Delta\boldsymbol{\varepsilon}^p = \mathbf{0}$ in V, i.e. plastic shakedown occurs under the load (P^0, β). Additionally, since the plastic strain rate history remains unaltered on application of δP^0, plastic shakedown also occurs under the loads $(P^0+\delta P^0, \beta)$ with arbitrary small δP^0 (but the mechanical part of δP^0 with application points upon V_e).

Conversely, let $(P^0, \beta) \in B_F$ and let $(P^0+\delta P^0, \beta) \in B_F$ for arbitrary small δP^0 and β taken as constant. Since, then, $\Delta\boldsymbol{\varepsilon}^p = \mathbf{0}$ and $\Delta\delta\boldsymbol{\varepsilon}^p = \mathbf{0}$ in V, from eqns (40) and (46) it follows that

$$\delta^2 W = \delta^2 A^0 = \int_V \delta\boldsymbol{\sigma}^0 : \Delta\delta\boldsymbol{\varepsilon}^p \, \mathrm{d}V = 0, \tag{52}$$

such that eqn (47) is met and consequently properties (b) and (c) above hold. In conclusion, the following proposition can be stated.

Proposition 3. In a body of generalized standard elastic–plastic material subjected to cyclic loads (P^0, β), if this load causes plastic shakedown, the related steady cycle is insensitive to (small) increments δP^0 of the permanent load, i.e. on application of the increment δP^0 the PAM $(\dot{\boldsymbol{\varepsilon}}^p, \dot{\boldsymbol{\xi}})$ remains unaltered, together with the stresses $\boldsymbol{\sigma}$ and the internal variables $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ within the region V_p where alternating plastic strains take place, and furthermore this region V_p (which cannot include the application points of the mechanical part of δP^0) also remains fixed.

The above reasoning changes somewhat in the case where $(P^0, \beta) \in \partial B_F$ because then δP^0 cannot be arbitrary. This point can be approached with arguments as in Polizzotto *et al.* (1990), but this is not pursued here for brevity.

The above proposition includes a particular result of Mróz (1972) for a P^0 representing an imposed displacement upon $\partial_u V$.

As a consequence of the above results, it can also be stated that the steady cycle of a load $(P^0, \beta) \in B_F$ is sensitive only to increments $\delta\beta$ of the time variable load, and that the steady cycle of a load within the ratchetting zone B_R is fully sensitive to load changes.

8. THE ELASTIC REGION V_e

The elastic region V_e is, by definition, that part of V where no plastic yielding occurs after the stabilization time, and thus where the plastic strains $\boldsymbol{\varepsilon}^p$ and the internal variables $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$, if not vanishing, are time-independent (see Proposition 2). The following proposition can be stated.

Proposition 4. In the steady cycle of an elastic–plastic structure, an elastic region V_e can exist if, and only if, the structure is in a condition of shakedown, either plastic shakedown (in which case $V_e \subset V$) or elastic shakedown (in which case $V_e = V$).

This statement was proved for perfect plasticity by Polizzotto (1993a) and also holds good in the present nonlinear hardening case. In order to prove this, it is obviously sufficient to consider only loads above the elastic shakedown limit load.

Let a load $(P^0, \beta) \notin B_S$ be considered together with the related steady cycle described by the state variables $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$, $\mathbf{u}$ .... The assumption is made that there exists a nonempty elastic region $V_e \subset V$ where no plastic yielding occurs in the steady cycle. Equations (22)–(27) and (32) are satisfied and in particular the plasticity conditions (23) read

$$\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0, \quad \lambda \geqslant 0, \quad \lambda\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant 0 \text{ in } V_p \times (0, \Delta t) \tag{53a}$$

$$\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) < 0, \quad \lambda = 0 \text{ in } V_e \times (0, \Delta t). \tag{53b}$$

Furthermore, let an additional permanent load δP^0 be applied upon V (but its mechanical part only upon V_e) and let, correspondingly, the time-independent stress field $\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^0 + \delta\boldsymbol{\rho}$ and the time-independent internal variable field $\delta\boldsymbol{\chi}$ be introduced. $\delta\boldsymbol{\sigma}^0$ is, by definition, the elastic stress response of V to δP^0, and $\delta\boldsymbol{\rho}$ is some time-independent self-stress field in V By hypothesis, δP^0 is sufficiently small and $\delta\boldsymbol{\rho}$, $\delta\boldsymbol{\chi}$ so chosen such as to satisfy the following conditions:

$$\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^0 + \delta\boldsymbol{\rho} = \mathbf{0}, \quad \delta\boldsymbol{\chi} = \mathbf{0} \text{ in } V_p \tag{54a}$$

$$\phi(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \boldsymbol{\chi} + \delta\boldsymbol{\chi}) < 0 \text{ in } V_e \times (0, \Delta t). \tag{54b}$$

With these conditions satisfied, it is easily recognized that eqns (22)–(27) and (32) are still satisfied with the same rate variables λ, $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\boldsymbol{\xi}}$, $\dot{\boldsymbol{\sigma}}^R$, $\dot{\mathbf{u}}^R$ and with the state variables substituted with new ones (denoted with asterisks), say $\boldsymbol{\sigma}^*$, $\boldsymbol{\rho}^*$, $\boldsymbol{\chi}^*$, $\boldsymbol{\xi}^*$, defined as

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma} + \delta\boldsymbol{\sigma} = \boldsymbol{\sigma}^{E*} + \boldsymbol{\rho}^* + \boldsymbol{\tau}, \quad \boldsymbol{\chi}^* = \boldsymbol{\chi} + \delta\boldsymbol{\chi} \tag{55a}$$

$$\boldsymbol{\sigma}^{E*} = \boldsymbol{\sigma}^E + \delta\boldsymbol{\sigma}^0, \quad \boldsymbol{\rho}^* = \boldsymbol{\rho} + \delta\boldsymbol{\rho}, \tag{55b}$$

but with the stress $\boldsymbol{\tau}(\mathbf{x}, t)$ left unchanged. The new equation set so restated provides the steady cycle pertaining to the load ($P^0 + \delta P^0$, β). Taking consideration of the uniqueness properties of the solution to the above equation set, it results that the steady cycles of (P^0, β) and ($P^0 + \delta P^0$, β) are characterized by the same rate variable histories, as well as by the same stress $\boldsymbol{\sigma}$ and internal variables $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ in V_p. As this fact remains true with δP^0 tending to vanish in an arbitrary way, it can be stated that the steady cycle of (P^0, β) manifests insensitivity to increments of the permanent load. This is sufficient to assert that, as a consequence, $\delta^2 W = 0$ for arbitrary small δP^0 and that plastic shakedown thus occurs under the load (P^0, β), with V_p being the region where alternating plastic strains occur. So the proof is complete.

Another property of the steady cycle is expressed by the following proposition.

Proposition 5. In a structure which is in a condition of plastic shakedown, its elastic part V_e finds itself in a condition of (partial) elastic shakedown.

For perfect plasticity, this property was given by Ponter (1983) without proof and subsequently by Polizzotto (1989, 1993a); see also Mróz (1972). In order to prove that Proposition 5 also holds in the present nonlinear hardening case, one has to remember that, in the steady cycle, the yield condition is not attained in V_e, i.e.

$$\phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) < 0 \text{ in } V_e \times (0, \Delta t), \tag{56}$$

where the stress $\boldsymbol{\sigma}$ is represented as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E(\mathbf{x}, t) + \boldsymbol{\rho}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}, t) \tag{57}$$

and the internal variable $\boldsymbol{\chi}$ is time-independent within V_e. Since both $\boldsymbol{\rho}$ and $\boldsymbol{\chi}$ are not uniquely determined within V_e, they can be cast as

$$\boldsymbol{\rho} = \hat{\boldsymbol{\rho}}(\mathbf{x}) + \mathbf{r}(\mathbf{x}), \quad \boldsymbol{\chi} = \mathbf{h}(\mathbf{x}) \text{ in } V_e, \tag{58}$$

where $\hat{\boldsymbol{\rho}}(\mathbf{x})$ is a suitable continuation within V_e of the uniquely determined field $\boldsymbol{\rho}$ in V_p, $\mathbf{r}(\mathbf{x})$ is a self-stress field in V_e and $\mathbf{h}(\mathbf{x})$ is some time-independent stress-like internal variable in V_e. With this choice, the stress $\boldsymbol{\sigma}$ can be written as

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}^E(\mathbf{x}, t) + \mathbf{r}(\mathbf{x}) \text{ in } V_e \times (0, \Delta t), \tag{59}$$

where

$$\hat{\boldsymbol{\sigma}}^E(\mathbf{x}, t) = \boldsymbol{\sigma}^E(\mathbf{x}, t) + \hat{\boldsymbol{\rho}}(\mathbf{x}) + \boldsymbol{\tau}(\mathbf{x}, t) \text{ in } V_e \times (0, \Delta t). \tag{60}$$

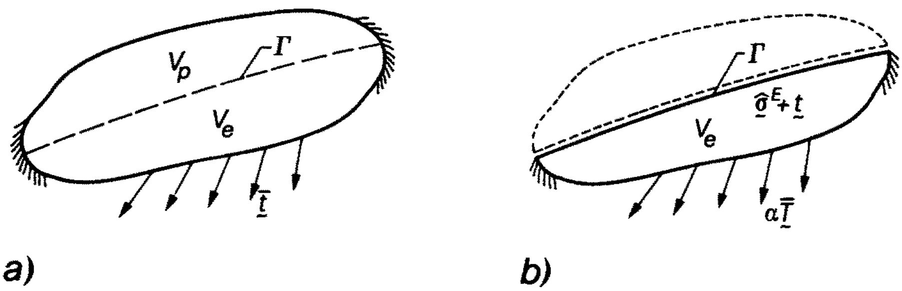

Fig. 2. Elastic–plastic solid in a state of plastic shakedown. (a) Elastic (V_e) and alternating plasticity (V_p) regions. (b) Isolated body V_e after removal of V_p, subjected to additional mechanical loads ($\alpha\bar{T}$) to be equilibrated by the stresses **t**.

However, the latter stresses $\hat{\boldsymbol{\sigma}}^E$ can be interpreted as the elastic stress response of the body V_e—considered isolated after the removal of V_p—to external actions as in fact (Fig. 2a, b):

$\boldsymbol{\sigma}^E(\mathbf{x}, t)$ is the elastic stress response of V_e to the loads directly applied upon it and to the tractions $\mathbf{t}^E_{(\nu)} = \boldsymbol{\sigma}^E \cdot \boldsymbol{\nu}$ at points of the interface Γ between V_e and V_p, $\boldsymbol{\nu}$ being the unit normal to Γ (external with respect to V_e);

$\hat{\boldsymbol{\rho}}(\mathbf{x})$ can always be considered as the elastic stress response of V_e to the tractions $\mathbf{t}^{(\rho)}_{(\nu)} = \hat{\boldsymbol{\rho}} \cdot \boldsymbol{\nu}$ on Γ, these tractions being unique on Γ like $\hat{\boldsymbol{\rho}}$;

$\boldsymbol{\tau}(\mathbf{x}, t)$, as the elastic stress response to the plastic strains that arise in V_p in the steady cycle, can also be identified as the elastic response to the tractions $\mathbf{t}^{(\tau)}_{(\nu)} = \boldsymbol{\tau} \cdot \boldsymbol{\nu}$ acting upon Γ.

As a consequence, one can affirm that there certainly exists, in the isolated body V_e, a time-independent self-stress field, $\mathbf{r}(\mathbf{x})$, and a time-independent stress-like internal variable field, $\mathbf{h}(\mathbf{x})$, such that the yield condition is violated nowhere, i.e. by eqns (56)–(60),

$$\phi(\hat{\boldsymbol{\sigma}}^E + \mathbf{r}, \mathbf{h}) < 0 \text{ in } V_e \times (0, \Delta t). \tag{61}$$

Then, according to the Bleich–Melan theorem generalized to nonlinear hardening (see Halphen, 1979; Maier, 1987; Polizzotto *et al.*, 1991), the above circumstance is sufficient for stating that elastic shakedown occurs in V_e.

It immediately follows that, as in perfect plasticity, plastic shakedown in V persists as long as the body V_e has the capacity to shake down in the elastic regime under the action of additional mechanical loads δP^0, and that the exhaustion of this capacity marks the transition from plastic shakedown to ratchetting. However, this point is not pursued here due to lack of space.

9. COMMENTS AND CONCLUSIONS

For a class of elastic–plastic hardening materials (generalized standard materials), the existence of a steady cycle response in a continuous structure subjected to given cyclic loads has been proven and discussed. Also, an equation set useful for the direct determination of such a response has been provided, and the characters of the steady cycle studied. Basically, in the steady state of the structure, the stresses $\boldsymbol{\sigma}$ and the internal variables $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$, as well as the plastic strain rates $\dot{\boldsymbol{\varepsilon}}^p$, turn out to be periodic with the same period of the applied loads. Furthermore, the sensitivity of the steady cycle to the load parameters (namely, the periodic load multiplier and permanent loads) has been studied showing that, as for perfectly plastic materials, the following properties hold true.

(1) At plastic shakedown, the body finds itself split into two regions, $V = V_p \cup V_e$, where V_p is the region of alternating plastic strains and V_e is the elastic region where a partial elastic shakedown condition is established.

(2) On applying upon V (small) increments of permanent loads—but the increments of the mechanical loads upon V_e—the plastic shakedown steady cycle response turns out to be insensitive to these load increments, in the sense that the state variables $\boldsymbol{\sigma}$, $\boldsymbol{\chi}$, $\boldsymbol{\xi}$ and the evolution variables $\dot{\boldsymbol{\varepsilon}}^p$, $\dot{\boldsymbol{\chi}}$ and $\dot{\boldsymbol{\xi}}$ remain unaltered in V_p, together with V_p itself, whereas some additional stresses arise in V_e, equilibrating the mechanical load increment.

(3) The transition from plastic shakedown to ratchetting is marked by a state of the body in which the elastic region V_e—viewed as an isolated body after the removal of V_p—exhausts its capacity of shaking down in the elastic regime under the action of additional permanent mechanical loads.

Though the above results hold under the simplifying hypotheses of small displacements and strains and of constitutive equations independent of temperature variations, they are of interest for structural design purposes, in particular within nuclear energy production plant applications. In the author's opinion, it is quite interesting to realize that concepts and methods already envisaged for perfectly plastic material models also hold good in the present, more realistic nonlinear hardening material models. The present paper is mainly concerned with the theoretical aspects of the steady cycle response problem; it provides a firm basis for further studies devoted to topics such as the following:

(a) The direct determination of the boundary between the plastic shakedown zone and the ratchetting zone—for perfectly plastic materials and for hardening materials with saturation—with procedures like those given by Ponter (1983), Ponter and Karadeniz (1985a, b), Ponter *et al.* (1990), and Polizzotto (1993a, b).

(b) The direct determination of the post-transient residual stresses—for non-isotropically hardening materials—by suitably extending to the present nonlinear hardening case the so-called Minimum Total Plastic Over-Potential Principle established by Polizzotto (1993b, c) for perfectly plastic materials.

(c) Formulation of criteria to predict the type of steady cycle produced by a given cyclic load, like the plastic shakedown theorems given by Polizzotto (1993b) for perfect plasticity.

Acknowledgements—This paper is part of a research project sponsored by the Ministero della Università e della Ricerca Scientifica e Tecnologica, MURST. The financial support is gratefully acknowledged.

REFERENCES

Ainsworth, R. A. (1977). Bounding solutions for creeping structures subjected to load variations above the shakedown limit. *Int. J. Solids Structures* **13**, 971–980.

Ainsworth, R. A., Goodall, I. W. and Walters, D. J. (1980). Cyclic loading in the creep range. *Proc. Int. Conf. on Engineering Aspects of Creep*, I. Mech. E/ASME, Sheffield, C215/80, pp. 153–160.

Dafalias, Y. F. and Popov, E. P. (1975). A model for nonlinearly hardening materials for complex loading. *Acta Mech.* **21**, 173–192.

Frederick, C. O. and Armstrong, P. J. (1966). Convergent internal stresses and steady cyclic state of stress. *J. Strain Anal.* **1**, 154–162.

Gokhfeld, D. A. and Cherniavsky, D. F. (1980). *Limit Analysis of Structures at Thermal Cycling.* Sijthoff & Noordhoff, Alphen aan der Rijn, The Netherlands.

Halphen, B. (1979). Steady cycles and shakedown in elastic–viscoplastic and plastic structures. In *Matériaux et Structures Sous Chargement Cyclique*, pp. 203–230. Association Amicale des Ingenieurs Anciens Eléves de E.N.P.C., Paris.

Halphen, B. and Nguyen, Q. S. (1975). Sur les matériaux standards généralisées. *J. Méc.* **14**, 39–63.

Krieg, R. D. (1975). A practical two surface plastic theory. *J. Appl. Mech.* **42**, 641–646.

Lemaitre, J. and Chaboche, J.-L. (1985). *Mécanique des Matériaux Solides.* Dunod, Paris.

Lubliner, J. (1990). *Plasticity Theory.* Macmillan, New York.

Maier, G. (1987). A generalization to nonlinear hardening of the first shakedown theorem for discrete elastic–plastic models. *Atti Accademia dei Lincei, Rend. Fis.* **8**, LXXX, 161–174.

Maier, G. and Novati, G. (1987). Deformation bounds for elastic–plastic discrete structures with piecewise linear yield laws and nonlinear hardening. In *Computational Plasticity* (Edited by D. R. J. Owen, E. Hinton and E. Oñate), pp. 17–31. Pineridge Press, Swansea, U.K.

Martin, J. B. (1975). *Plasticity: Fundamentals and General Results.* MIT Press, Cambridge, MA.

Mróz, Z. (1969). An attempt to describe the behavior of metals under cyclic loads using a more general work hardening model. *Acta Mech.* **7**, 199–212.

Mróz, Z. (1972). On the theory of steady plastic cycles in structures. *Proc. First Int. Conf. on Structural Mechanics in Reactor Technology*, Vol. 6, Part L, pp. 489–501.

Polizzotto, C. (1993a). A study on plastic shakedown of structures. Part I: basic properties. *J. Appl. Mech.* **60,** 318–323.

Polizzotto, C. (1993b). A study on plastic shakedown of structures. Part II: theorems. *J. Appl. Mech.* **60,** 324–330.

Polizzotto, C. (1993c). The assessment of the residual post-transient stresses. *Rend. Mat. Acc. Lincei* S.9, **4,** 133–144.

Polizzotto, C., Borino, G. and Fuschi, P. (1990). On the steady-state response of elastic-perfectly plastic solids to cyclic loads. In *Inelastic Solids and Structures* (Edited by M. Kleiber and J. A. König), pp. 473–488. Pineridge Press, Swansea, U.K.

Polizzotto, C., Borino, G., Caddemi, S. and Fuschi, P. (1991). Shakedown problems for material models with internal variables. *Eur. J. Mech. A. Solids* **10,** 621–639.

Ponter, A. R. S. (1972). Deformation, displacement and work bounds for structure in a state of creep and subjected to variable loading. *J. Appl. Mech.* **39,** 953–958.

Ponter, A. R. S. (1983). Shakedown and ratchetting below the creep range. Report EUR 8702 EN, Commission of European Communities, Brussels.

Ponter, A. R. S. and Karadeniz, S. (1985a). An extended shakedown theory for structures that suffer cyclic thermal loadings, Part 1: theory. *J. Appl. Mech.* **52,** 877–882.

Ponter, A. R. S. and Karadeniz, S. (1985b). An extended shakedown theory for structures that suffer cyclic thermal loadings, Part 2: applications. *J. Appl. Mech.* **52,** 883–889.

Ponter, A. R. S., Karadeniz, S. and Carter, K. F. (1990). Extended upper bound shakedown theory and finite element method for axisymmetric thin shells. In *Inelastic Solids and Structures* (Edited by M. Kleiber and J. A. König), pp. 433–449. Pineridge Press, Swansea, U.K.